INSTITUTE FOR MATHEMATICS
STRUCTURES CLUSTER OF EXCELLENCE
UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Michael Bleher
*Institute for Mathematics, Heidelberg University*

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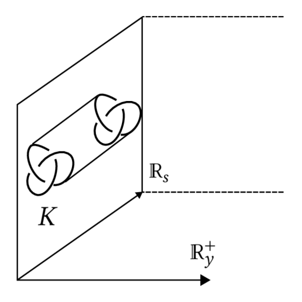
# HAYDYS-WITTEN INSTANTONS

## *AND THE GAUGE THEORETIC APPROACH TO KHOVANOV HOMOLOGY*



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based on PhD Thesis (doi:10.11588/HEIDOK.00034010) and arxiv:2307.15056.

# Motivation

**Topological Field Theories give rise to topological invariants.**

i. *Chern-Simons theory*

$$Z_{\mathrm{CS}}(X^3) \rightsquigarrow \text{Witten-Reshetikhin-Turaev invariants}$$

$$Z_{\mathrm{CS}}(S^3, K) \rightsquigarrow \text{Jones polynomial}$$

ii. *topologically twisted $d = 4$ $\mathcal{N} = 2$ super Yang-Mills theory*

$$Z^Q_{\mathrm{SYM}}(W^4; \underbrace{\gamma_1, \dots, \gamma_n}_{\in H_\bullet(W^4)}) \rightsquigarrow \text{Donaldson polynomials}$$

# Motivation

iii *topologically twisted $d = 4$ $\mathcal{N} = 2$ super Yang-Mills theory*

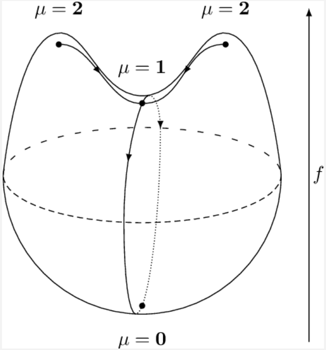
- on $W^4 = \mathbb{R}_s \times X^3$
- "coupled" to Chern-Simons theory @ $s \to \pm\infty$

## classical states

critical points of Chern-Simons action = flat $G$-connections on $X^3$

## quantum corrections

gradient flow of Chern-Simons action = Yang-Mills instantons on $\mathbb{R}_s \times X^3$ (ASD $G$-connections)




$\rightsquigarrow$ $HF^\bullet(X^3)$ *Yang-Mills* Instanton Floer theory

# Motivation – by analogy

(iv) *topologically twisted* $d = 5$ $\mathcal{N} = 2$ *super Yang-Mills theory*

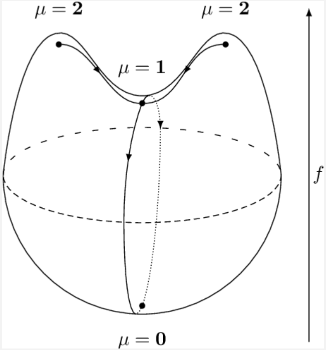
- on $M^5 = \mathbb{R}_s \times W^4$
- "coupled" to top. tw. $d = 4$ $\mathcal{N} = 2$ SYM @ $s \to \pm\infty$

## classical states

critical points of super Yang-Mills action = Kapustin-Witten soln. on $W^4$ (phase-shifted ASD $G_{\mathbb{C}}$-connections)

## quantum corrections

gradient flow of Kapustin-Witten equations = Haydys-Witten instantons on $\mathbb{R}_s \times W^4$




$\rightsquigarrow$ $HF^\bullet(W^4)$ *Haydys-Witten* Instanton Floer theory

# Haydys-Witten instanton Floer Theory

$E \to M^5$ $G$-principal bundle

$(M^5, g)$ Riemannian 5-manifold

$v$ nowhere vanishing unit vector field

$A$ connection one-form

$B$ adjoint-valued self-dual 2-form

## (anti-)self-dual 2-forms in 5d

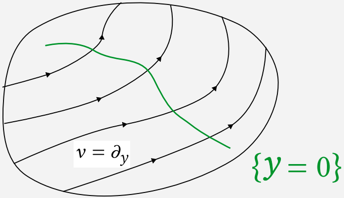
Let $\eta := g(v, \cdot) \in \Omega^1(M^5)$ be the 1-form dual to $v$.
Then $*_5(\cdot \wedge \eta)$ induces **eigenvalue decomposition**

$$\Omega^2(M^5) = \Omega^2_{v,+} \oplus \Omega^2_{v,0} \oplus \Omega^2_{v,-}$$

| | | | | |
|---|---|---|---|---|
| rank | 10 | 3 | 4 | 3 |

*Remark:*
*$\Omega^2_{v,+}$ is a lift of $4d$ self-dual forms, where $v$ determines which direction is "additional" in $5d$.*

$U \simeq W^4 \times \mathbb{R}_y$




$\Omega^2_{\partial_y,\pm}(W^4 \times \mathbb{R}_y) \simeq \Omega^2_\pm(W^4)$

## cross-product on $\Omega^2_{v,+}(M, \mathrm{ad}\, E)$

- cross-product $(\cdot \times \cdot)$ on $\Omega^2_{v,+}$ (fiber $\mathbb{R}^3$)
- Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ on $\mathrm{ad}\, E$ (fiber $\mathfrak{g}$)

$\rightsquigarrow$ yield bilinear map $\sigma$ on $\Omega^2_{v,+}(M^5, \mathrm{ad}\, E)$ (fiber $\mathbb{R}^3 \otimes \mathfrak{g}$):

$$\sigma(\cdot,\cdot) := (\cdot \times \cdot) \otimes [\cdot,\cdot]_{\mathfrak{g}}$$

## codifferential on $\Omega^2_{v,+}(M, \mathrm{ad}\, E)$

$$\delta_A^+ : \Omega^2_{v,+}(M^5, \mathrm{ad}\, E) \xrightarrow{\nabla^{A,LC}} T^*M \otimes \Omega^2_{v,+}(M, \mathrm{ad}\, E) \xrightarrow{\text{contr}} \Omega^1(M^5, \mathrm{ad}\, E)$$

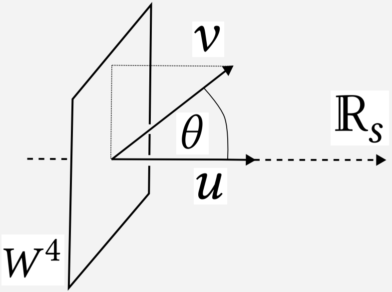
Let $A \in \mathcal{A}(E)$ connection one-form, $B \in \Omega^2_{v,+}(M^5, \mathrm{ad}\, E)$ self-dual two-form.

### Haydys-Witten equations

$$F_A^+ = \sigma(B,B) + \nabla_v^A B$$
$$\iota_v F_A = \delta_A^+ B$$

Now consider 5-manifolds of the form $M^5 = \mathbb{R}_s \times W^4$.
**boundary condition** @ $s \to \pm\infty$ (cylindrical ends): **asymptotically stationary** solutions.

Assume **incidence angle** $g(\partial_s, v) = \cos\theta$ is constant.




## $\mathbb{R}_s$-invariant Haydys-Witten equations

$$\mathrm{HW}_v(A,B) \rightsquigarrow \begin{cases} \mathrm{VW}(\tilde{A}, B, A_s) & \theta = 0 \pmod{\pi} \\ \mathrm{KW}_\theta(\tilde{A}, \phi) & \text{else} \end{cases}$$

$\rightsquigarrow$ Haydys-Witten solutions on $\mathbb{R}_s \times W^4$ **interpolate** between Vafa-Witten / $\theta$-Kapustin-Witten solutions on $W^4$ at $s \to \pm\infty$.

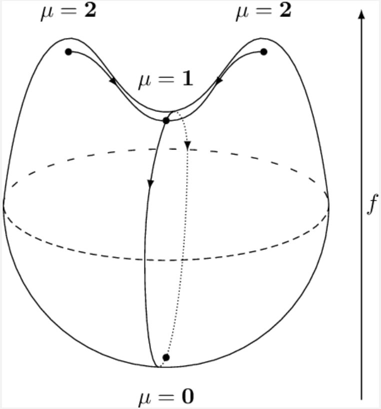
## Floer chains

$$CF_\theta(W^4) := \bigoplus_{x \in \mathcal{M}^{\mathrm{KW}_\theta}(W^4)} \mathbb{Z} \cdot \langle x \rangle$$

## Floer differential

$$\mathcal{M}_v(x,y) = \left\{ \begin{array}{c} (A,B) \in \mathcal{M}^{\mathrm{HW}_v}(\mathbb{R}_s \times W^4), \\ \lim_{s\to-\infty}(A,B) = x, \\ \lim_{s\to\infty}(A,B) = y \end{array} \right\}$$

$$d_v\langle x \rangle := \sum_{\mu(x,y)=1} \#\mathcal{M}_v(x,y)/\mathbb{R} \cdot \langle y \rangle$$




## ⇝ HW-instanton Floer cohomology

$$HF^\bullet_\theta(W^4) := H^\bullet(CF_\theta(W^4), d_v)$$

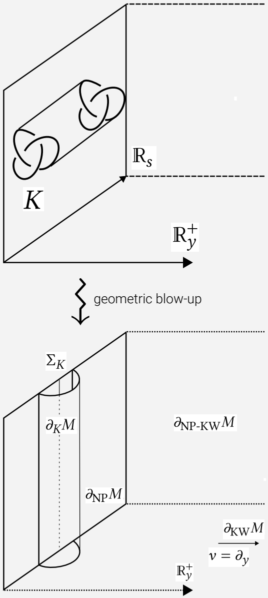
# Towards Khovanov Homology

## Where are the knots?

Following Witten, physical theory suggests to consider

$$M^5 = \mathbb{R}_s \times \underbrace{X^3 \times \mathbb{R}_y^+}_{W^4}, \quad v = \cos\theta\partial_s + \sin\theta\partial_y,$$

together with a knot $K \subset X^3 = \partial W^4$.



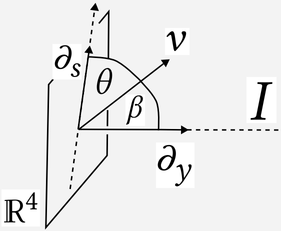

## How to include the knot?

Geometric blow-up along $\Sigma_K = \mathbb{R}_s \times K \times \{y = 0\}$ and specify boundary behaviour:

- **Nahm pole** at original boundary ($y = 0$),
- and **monopole-like singularity** at blown-up boundary ($R = 0$).

**boundary conditions** $@y \to 0$ (original boundary): **locally boundary-independent** solutions.

Assume **incidence angle** $g(v, \partial_y) = \cos\beta$ is constant.




## $\mathbb{R}^4$-invariant Haydys-Witten equations

$$\mathrm{HW}_v(A, B) \rightsquigarrow \mathrm{NP}^{\mathbb{O}}_{\beta}(\tilde{A}, \Phi)$$

## Nahm pole boundary conditions

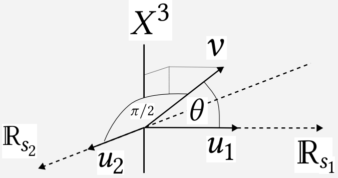
Modelled on solution of ($\beta$-twisted octonionic) Nahm's equation with pole at $y = 0$.

$$A_i = \sin\beta \frac{t_i^\tau}{y} + O(y^{-1+\epsilon}) \qquad B_i = \cos\beta \frac{t_i}{y} + O(y^{-1+\epsilon}) \qquad A_s, A_y = 0 + O(y^{-1+\epsilon})$$

$\{t_i\}_{i=1,2,3}$ $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.

**boundary conditions** $@R \to 0$ (blown-up boundary): **locally boundary-independent** solutions.

Assume **glancing angle** $g(v, \partial_s) = \cos\theta$ is constant.




## $\mathbb{R}^2$-invariant Haydys-Witten equations

$$\mathrm{HW}_v(A, B) \rightsquigarrow \mathrm{TEBE}_\theta(\tilde{A}, \phi, c_1, c_2)$$

## Knot singularity boundary conditions

Modelled on monopole solution of ($\theta$-twisted) extended Bogomolny equation with 'magnetic charge' $\lambda \in \Gamma^\vee_{\mathrm{char}}$.

$$A = A^{\lambda,\theta} + O(R^{-1+\epsilon}) \qquad B = B^{\lambda,\theta} + O(R^{-1+\epsilon})$$

Physics: For $X^3 = S^3$ or $\mathbb{R}^3$ and

- $v = \partial_y$ ($\implies \theta = \pi/2, \beta = 0$)
- $(A, B)$ asymptotically stationary at cylindrical ends ($s \to \pm\infty, y \to \infty$)
- $(A, B)$ satisfy Nahm pole and knot singularity BCs at boundaries ($y \to 0, R \to 0$)

**Conjecture (Witten 2011)**

$$HF_{\pi/2}\big([S^3; K] \times \mathbb{R}^+_y\big) = \mathrm{Kh}^{\bullet,\bullet}(K)$$

Q: How to test this?

Problem: Solutions to Haydys-Witten and Kapustin-Witten equations are not well-understood.

# Decoupled Haydys-Witten Equations

- $M^5 = \mathbb{R}_s \times X^3 \times \mathbb{R}_y^+$
- $v = \sin\theta \partial_s + \cos\theta \partial_y$

$\Longrightarrow \ker g(v, .) \simeq T(\mathbb{R} \times X^3)$ admits an almost Hermitian structure $J$.

$J$ lifts to $J \otimes J \circlearrowright \Omega^2_{v,+}(M^5)$ with eigenvalues $\{+1, -1, -1\}$.
Write $J^\pm := (1 \pm J \otimes J)/2$ for the projections.

## Definition

$$F_A^+ = J^+(\sigma(B,B) + \nabla_v^A B) \qquad 0 = J^-(\sigma(B,B) + \nabla_v^A B)$$
$$\imath_v F_A = \delta_A^+ J^+ B \qquad 0 = \delta_A^+ J^- B$$

*Remark: Contributions from $F_A$ and $B$ in the negative eigenspace of $J \otimes J$ are "decoupled".*

## Hermitian Yang-Mills structure

On local holomorphic patch $(w = s + ix^1, z = x^2 + ix^3, y)$ of $M^5 = \mathbb{R}_s \times X^3 \times \mathbb{R}^+_y$:

$$\mathcal{D}_0 = \nabla^A_{\bar{w}} \qquad \mathcal{D}_1 = \nabla^A_{\bar{z}} \qquad \mathcal{D}_2 = \nabla^A_y - i[B_1, \cdot] \qquad \mathcal{D}_3 = [B_2, \cdot] + i[B_3, \cdot]$$

Then Haydys-Witten equations and their decoupled version are

$$HW_v(A,B) = 0 \iff \left\{ \begin{matrix} \overline{[\mathcal{D}_0, \mathcal{D}_i]} + \frac{1}{2}\epsilon_{ijk}[\mathcal{D}_j, \mathcal{D}_k] = 0 \\ \sum_{\mu=0}^{3}[\bar{\mathcal{D}}_\mu, \mathcal{D}_\mu] = 0 \end{matrix} \right\}$$

$$\Uparrow$$

$$dHW_{v,J}(A,B) = 0 \iff \left\{ \begin{matrix} [\mathcal{D}_\mu, \mathcal{D}_\nu] = 0 \\ \sum_{\mu=0}^{3}[\bar{\mathcal{D}}_\mu, \mathcal{D}_\mu] = 0 \end{matrix} \right\} \longleftarrow G_{\mathbb{C}} - \text{invariant! Use ideas of DUY}$$

There is a Weitzenböck formula

$$\int_{M^5} \|\mathrm{HW}_v(A,B)\|^2 = \int_{M^5} \|\mathrm{dHW}_{v,J}(A,B)\|^2 + \int_{M^5} d\chi$$

## Theorem (B. '23)

Let $M^5 = \mathbb{R}_s \times X^3 \times \mathbb{R}_y^+$ and $v = \partial_y$. Assume

- $\mathbb{R}_s \times X^3$ is ALE or ALF gravitational instanton, and Nagy-Oliveira's Conjecture holds
- $(A, B)$ satisfy corresponding BCs (Nahm poles, knot singularities, $\theta$-Kapustin-Witten asymptotics)

Then $\int_{M^5} d\chi \to 0$.

## Corollary

Under the assumptions of the theorem $\mathrm{HW}_v(A,B) = 0 \iff \mathrm{dHW}_{v,J}(A,B) = 0$.
In particular: $HF^{\bullet}_{\pi/2}([\mathbb{R}^3; K])$ is fully determined by "decoupled" Haydys-Witten instantons.

## proof idea

regularize $\int_{M^5} d\chi \rightsquigarrow \sum_i \int_{\partial_i M^5} \chi$ by a compact exhaustion of $M^5$ (respecting incidence angles of $v$).

@ $y \to 0$:

- elliptic regularity of $\mathrm{HW}_\theta \implies (A, B)$ polyhomogeneous (∃ asymptotic series in $y^\alpha (\log y)^k$)
- expand $\chi$ around Nahm pole and knot singularities

$$\rightsquigarrow \ \chi = \begin{cases} O(y^\epsilon)\,\mathrm{vol}_{\partial_{NP} M} & (y \to 0) \\ O(R^{-1+\epsilon})\,\mathrm{vol}_{\partial_K M} & (R \to 0) \end{cases}$$

@ $y \to \infty$:

**Conj** (Nagy-Oliveira '21, B. '23)
$W^4$ ALE or ALF, $\mathrm{KW}_\theta(A, \phi) = 0$, finite energy $\implies A$ flat, $\nabla^A \phi = 0 = [\phi \wedge \phi]$.

$$\rightsquigarrow \ \chi \propto \nabla^A \phi \ \& \ [\phi \wedge \phi] \to 0.$$

@ $s \to \pm\infty$: mixture of both arguments

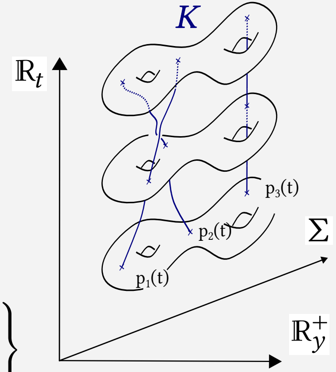
# Outlook / Future Research

Generalize He-Mazzeo's classification of Nahm pole solutions with $S^1$-invariant knot $K = S^1 \times \underbrace{\sqcup\{p_i\}}_{D} \subset S^1 \times \Sigma = X^3$

$$\mathcal{M}_K^{\mathrm{KW}} = \mathcal{M}_D^{\mathrm{EBE}} = \left\{ \begin{matrix} \mathcal{D}_0 = 0,\ [\mathcal{D}_i, \mathcal{D}_j] = 0 \\ \sum_{i=1}^3 [\bar{\mathcal{D}}_i, \mathcal{D}_i] = 0 \end{matrix} \right\} \overset{1:1}{\leftrightarrow} \left\{ \begin{matrix} \text{Higgs bundles w/} \\ \text{extra structure } @K \end{matrix} \right\}$$

to $S^1$-*dependent* knots $K = \mathbb{R}_t \times \sqcup\{p_i(t)\} \subset \mathbb{R}_t \times \Sigma = X^3$.

$$\mathcal{M}_{\Sigma_K}^{\mathrm{dHW}} = \left\{ \begin{matrix} [\mathcal{D}_\mu, \mathcal{D}_\nu] = 0 \\ \sum_{\mu=0}^3 [\bar{\mathcal{D}}_\mu, \mathcal{D}_\mu] = 0 \end{matrix} \right\} \overset{1:1}{\leftrightarrow} \left\{ \begin{matrix} \text{pseudo-holomorphic discs in} \\ \text{moduli space of Higgs bundles} \\ \text{w/ extra structure } @\Sigma_K \end{matrix} \right\}$$




Seems to lead to something that looks a lot like **symplectic Khovanov homology**?

# Thank you for your attention!