Haydys-Witten Instantons in the Gauge Theoretic Approach to Khovanov Homology

Mittwoch, 29. November 2023 19:36

Plan

(1) Motivation (Insights from physics)

(2) HW Floer Theory

(3) Khovanov Homology

(4) decoupled HW equations

(5) Outlook

(1) Motivation

Topological Field Theories give rise to topological invariants.

Famous Examples

(i) Chern-Simons Theory

$$Z_{CS}(S^3, K) \leftrightarrow \text{Jones Polynomial}$$

$$Z_{CS}(X^3) \leftrightarrow \text{Reshetikhin-Turaev}$$

(ii) topologically twisted $d=4$ $N=2$ susy Yang-Mills

$$Z^{Q}_{SYM}(W^4; \underbrace{\gamma_1, \ldots, \gamma_n}_{\in H_\bullet(W^4)}) \leftrightarrow \text{Donaldson Polynomials}$$

(iii) Hilbert space of states in

"top. tw. $d=4$ $N=2$ SYM on $\mathbb{R}_s \times X^3$
coupled to CS @ $s \to \pm\infty$"

classical states: critical pts of CS-functional
$=$ flat gauge connections on X^3

quantum corrections: gradient flow of CS-functional
$=$ ASD connections on $\mathbb{R}_s \times X^3$

$\leftrightarrow$ $HF^\bullet(X^3)$ YM-instanton Floer Theory

Analogously

"top. tw. $d=5$ $N=2$ SYM on $\mathbb{R}_s \times W^4$
coupled to top. tw. $d=4$ $N=4$ SYM @ $s \to \pm\infty$"

classical states: Solutions of KW-eqns on W^4

quantum corrections: Flow equations of KW-operator
$=$ HW-eqns on $\mathbb{R}_s \times W^4$

$\leftrightarrow$ $HF^\bullet(W^4)$ HW-instanton Floer Theory

Physics: This is a topological invariant of W^4

(2) HW-instanton Floer Theory

Start w/ HW-equations

(M^5, g) Riem. Mfld

v nowhere vanishing vector field

$E \to M$ G-principal bundle

A gauge connection

Let $\eta := g(v, \cdot) \in \Omega^1(M^5)$.

Then $*_5(\cdot \wedge \eta)$ induces eigenvalue decomposition

$$\Omega^2(M^5) = \Omega^2_{v,+} \oplus \Omega^2_{v,0} \oplus \Omega^2_{v,-}$$

rank 10 3 4 3

<u>NB</u>: $\Omega^2_{v,+}$ is a lift of 4d self-dual forms into 5d where v determines which direction is "additional". There is an iso (induced by projection)

$$\Omega^2_{\partial_y,+}(W^4 \times \mathbb{R}_y)\big|_y \cong \Omega^2_+(W^4)$$

There is a cross-product on $\Omega^2_{v,+} \cong \mathbb{R}^3$. Use this to define for elements of $\Omega^2_{v,+}(M, \mathrm{ad}\, E) \overset{\text{loc.}}{\simeq} \Omega^2_{v,+} \otimes \mathfrak{g}$

$$\sigma : \Omega^2_{v,+}(M^5, \mathrm{ad}\, E) \to \Omega^2_{v,+}(M^5, \mathrm{ad}\, E)$$

$$\sigma(\cdot, \cdot) := (\cdot \times \cdot) \otimes [\cdot, \cdot]_{\mathfrak{g}}$$

Eg. for $B = \sum_{i=1}^{3} e_i \otimes \phi_i \in \Omega^2_{v,+}$, then

$$\sigma(B, B) = e_1 \otimes [\phi_2, \phi_3] + (\text{cycl.})$$

$$\left(= g^{\sigma\tau} [B_{\mu\sigma}, B_{\nu\tau}]\right)$$

Also need codifferential

$$\delta^+_A : \Omega^2_{v,+}(M, \mathrm{ad} E) \xrightarrow{\nabla^{A,LC}} T^*M \otimes \Omega^2_{v,+}(M, \mathrm{ad} E) \xrightarrow[\text{contr.}]{\text{dualize}} \Omega^1(M, \mathrm{ad} E)$$

$$\omega \mapsto \nabla^{A,LC}_{\bullet} \otimes \omega \mapsto -\sum_{\mu} \iota_{\mu}(\nabla^{A,LC}_{\mu} \omega)$$

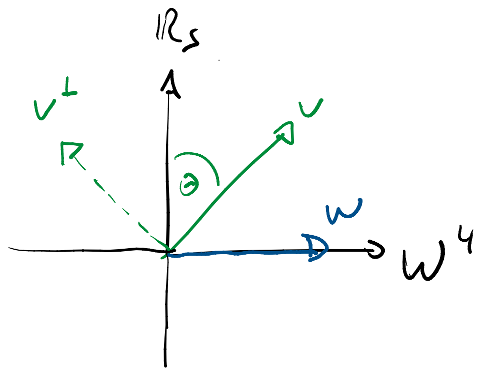
<u>Haydys-Witten eqns</u> $A \in \mathcal{A}(M^5)$, $B \in \Omega^2_{v,+}(M^5, \mathrm{ad}\, E)$

$$\left\{ \begin{aligned} F^+_A - \sigma(B, B) - \nabla^A_v B &= 0 \\ \iota_v F_A - \delta^+_A B &= 0 \end{aligned} \right\} \quad HW_v(A, B)$$

NB: For $B = 0$ these are anti-self duality equations in 5d.

Consider now $M^5 = \mathbb{R}_s \times W^4$

and assume $g(\partial_s, v) = \cos\theta$ is constant



Then can ask about $\underline{\mathbb{R}\text{-inv. HW-eqns}}$:

$\underline{\theta = 0}$ i.e. $u = v$ are parallel

$$A = A_p \, dp + \tilde{A} \quad , \quad B \in \Omega^2_{v,+}(M^5) \simeq \Omega^2_+(W^4)$$

$$A_p \cong C \in \Omega^2(W^4, \mathrm{ad}E), \qquad \tilde{A} \in \mathcal{A}(W^4)$$

$$\left\{ \begin{aligned} F_A^+ - \tfrac{1}{2}\sigma(B,B) - \tfrac{1}{2}[C,B] &= 0 \\ d_A^* B + d_A C &= 0 \end{aligned} \right\} \quad \mathrm{VW}(A,B,C)$$

$\underline{\Theta \neq 0}$

in coordinates $(p, q, x^a)_{a=1,2,3}$ of $\mathbb{R} \times W^4$

$u = \partial_p$, $v = \cos\Theta\, \partial_p + \sin\Theta\, \partial_q$

$$A = A_p\, dp + \tilde{A} \qquad \rightsquigarrow \qquad \tilde{A} \in \mathcal{A}(W^4)$$
$$B = \sum_{i=1}^{3} \phi_i e_i \qquad\qquad \phi = A_p\, dq + \phi_a\, dx^a \in \Omega^1(W^4, \mathrm{ad}\, E)$$

$$\left\{ \begin{aligned} \left[\cos\tfrac{\Theta}{2}\left(F_A - [\phi\wedge\phi]\right) - \sin\tfrac{\Theta}{2}\, d_A\phi\right]^+ &= 0 \\ \left[\sin\tfrac{\Theta}{2}\left(F_A - [\phi\wedge\phi]\right) + \cos\tfrac{\Theta}{2}\, d_A\phi\right]^- &= 0 \\ d_A^{*_4}\phi &= 0 \end{aligned} \right\} \quad KW_\Theta(A, \phi)$$

$\Theta \in [0, \pi]$

$\underline{\text{Rmk}}$ non-standard normalization $\Theta_{GU} = \frac{\Theta}{2}$.

"The" KW-eqns are $\pi/2$-KW equations (instead of $\pi/4$)

$\underline{\text{In summary}}$

$$HW_v(A, B) \xrightarrow{\mathbb{R}_s\text{-inv.}} \left\{ \begin{array}{ll} VW(i^*A, i^*B, C = A_s) & \Theta \equiv 0 \ (\mathrm{mod}\ \pi) \\ KW_\Theta(i^*A, \phi = A_s w^\flat + \iota_v B) & \text{else} \end{array} \right.$$

Put differently, HW-eqns on $\mathbb{R}_s \times W^4$ can be viewed as flow eqns/instantons between VW/KW-solutions.

$\rightsquigarrow$ HW-instanton Floer theory

Consider W^4 w/ non-vanishing unit v.f. w.

Floer chains

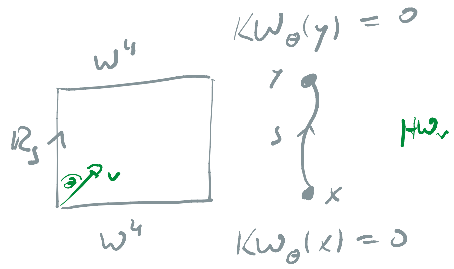
$$CF_\theta(W^4) := \bigoplus_{x \in \mathcal{M}^{KW}(W^4;\theta)} \mathbb{Z}\,|x\rangle$$

Floer differential

$$\hat{\mathcal{M}}(x,y) := \left\{ \begin{array}{l} (A,B) \in \mathcal{M}^{HW}\left(\mathbb{R}_s \times W^4,\ v = \sin\theta\,\partial_s + \cos\theta\, w\right) \\ \text{s.t. } (A,B) \longrightarrow \begin{array}{ll} x & s \to -\infty \\ y & s \to +\infty \end{array} \end{array} \right\}$$

& relative index $\mu(x,y) = \operatorname{ind} HW$

$$\rightsquigarrow \quad d_v\, x := \sum_{\mu(x,y)=1} \dim \hat{\mathcal{M}}(x,y)/\mathbb{R}\ |y\rangle$$



HW-instanton Floer cohomology

$$HF_\theta(W^4) := H\left(CF_\theta(W^4),\ d_v\right)$$

NB: This is expected to be functorial wrt. cobordisms.
In particular there are maps $HF_{\theta_1}(W^4) \to HF_{\theta_2}(W^4)$
induced by $v = \cos\theta\,\partial_s + \sin\theta\, w$ w/ $\theta(s = \pm\infty) = \theta_{1/2}$

(3) Khovanov Homology (where are the knots?)

Following Witten, consider

$$M^5 = \mathbb{R}_s \times \underbrace{X^3 \times \mathbb{R}_y^+}_{W^4 \text{ mfld w/ bdry}}$$

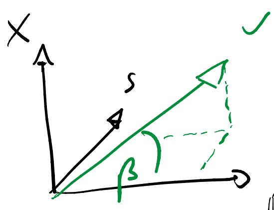
$v = \partial_y$ (later also deformations $v = \cos\theta\, \partial_s + \sin\theta\, \partial_y$)

$K \subseteq \partial W^4 = X^3$

and use Nahm pole boundary conditions with monopole-like knot singularity near K.

Nahm pole BC

Denote image of standard sl_2-triple $(e,h,f) \in \mathfrak{g}$ by t_i, $i=1,2,3$



$$A_i \sim \sin\beta\, \frac{t_i}{y} + \mathcal{O}(y^{-1+\varepsilon})$$

$$B_i \sim \cos\beta\, \frac{t_i}{y} + \mathcal{O}(y^{-1+\varepsilon})$$

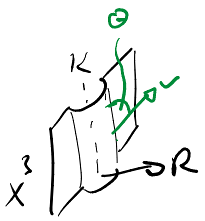
$$A_s, A_y \sim 0 + \mathcal{O}(y^{-1+\varepsilon})$$

$R_s \times X^3$-independent model solution of HW-eqns

Knot singularity BC

'magnetic charge' of K determined $\lambda \in \Gamma^\vee_{char}$.

Blow up along K $\rightsquigarrow$ $[X^3; K]$ mfld w/ corners



$$A \sim A^{\lambda,\theta} + \mathcal{O}(R^{-1+\varepsilon})$$

$$B \sim B^{\lambda,\theta} + \mathcal{O}(R^{-1+\varepsilon})$$

$\mathbb{R}_s \times K$ - independent model solutions

$$A_\vartheta = -(\lambda+1)\cos^2\psi \frac{(1+\cos\psi)^\lambda - (1-\cos\psi)^\lambda}{(1+\cos\psi)^{\lambda+1} - (1-\cos\psi)^{\lambda+1}} H$$

$$\phi_1 = -\frac{\lambda+1}{R} \frac{(1+\cos\psi)^{\lambda+1} + (1-\cos\psi)^{\lambda+1}}{(1+\cos\psi)^{\lambda+1} - (1-\cos\psi)^{\lambda+1}} H$$

$$\varphi = \frac{(\lambda+1)}{R} \frac{\sin^\lambda\psi \; \exp(i\lambda\vartheta)}{(1+\cos\psi)^{\lambda+1} - (1-\cos\psi)^{\lambda+1}} E$$

$$A_s = A_t = A_R = A_\psi = 0$$

Physics: For $X^3 = S^3$ or $\mathbb{R}^3$ w/

- NPK boundary conditions @ $y=0$
- (A,B) asymptotically stationary @ non-compact ends & finite energy

Conjecture (Witten 2011)

$$HF_{\pi/2}([\mathbb{R}^3; K]) = Kh^{\bullet,\bullet}(K)$$

geometric blowup

non-obvious bigrading

(3) decoupled Haydys-Witten equations

If $\quad M^5 = \mathbb{R}_s \times X^3 \times \mathbb{R}^+_y$, $v = \sin\theta\, \partial_s + \cos\theta\, \partial_y$

Then $\ker g(v,\cdot) = T(\mathbb{R} \times X^3)$ admits an almost Hermitian structure J , i.e. $J^2 = -id$, $g(J\cdot, J\cdot) = g(\cdot,\cdot)$

Rmk since $\mathbb{R}_s \times X^3$ is open & orientable $\Rightarrow$ ...

J lifts to $J \otimes J \circlearrowleft \Omega^2_{v,+}(M)$ w/ e.v. $\{+1, -1, -1\}$

locally
$$
\begin{array}{ll}
ds \wedge dx^1 + dx^2 \wedge dx^3 & +1 \\
ds \wedge dx^2 + dx^3 \wedge dx^1 & -1 \\
ds \wedge dx^3 + dx^1 \wedge dx^2 & -1
\end{array}
$$

Write $J^\pm = \frac{1}{2}(1 \pm J)$ for projection operators

Def. "Decoupled" Haydys-Witten equs

$$
\left\{
\begin{aligned}
F_A^+ &= J^+(\sigma(B,B) + \nabla_v^A B) \\
\iota_v F_A &= \delta_A^+ J^+ B \\
0 &= J^-(\sigma(B,B) + \nabla_v^A B) \\
0 &= \delta_A^+ J^- B
\end{aligned}
\right\} \quad DHW_{v,J}(A,B)
$$

These equations admit a Hermitian Yang-Mills structure!

Use (A, B) and J to define on a local patch (w, z, y)

$$\mathcal{D}_0 = \nabla^A_{\bar{w}} \qquad \mathcal{D}_2 = \nabla^A_y - i[\phi_1, \cdot]$$

$$\mathcal{D}_1 = \nabla^A_{\bar{z}} \qquad \mathcal{D}_3 = [\phi_2, \cdot] - i[\phi_3, \cdot]$$

Then

$$HW_v \left\{ \begin{array}{ll} \overline{[\mathcal{D}_0, \mathcal{D}_i]} + \frac{1}{2}\varepsilon_{ijk}[\mathcal{D}_j, \mathcal{D}_k] = 0 \\ \sum [\bar{\mathcal{D}}_\mu, \mathcal{D}_\mu] = 0 \end{array} \right\}$$

$$\Uparrow$$

$$DHW \left\{ \begin{array}{l} [\mathcal{D}_\mu, \mathcal{D}_\nu] = 0 \\ \sum [\bar{\mathcal{D}}_\mu, \mathcal{D}_\mu] = 0 \end{array} \right\}$$

$G_{\mathbb{C}}$-invariant

use ideas of DUY!

<u>NB</u>: This means that (at least some) HW-instantons might be more readily available.

In fact, there is a Weitzenböck formula

$$\int_{M^5} \| HW_v(A, B) \|^2 = \int_{M^5} \| DHW_{v,J} \|^2 + \int_{M^5} d\chi$$

$$\chi_1 = -2\,\mathrm{Tr}\,(F_A \wedge J^- B)$$
$$\chi_2 = -2\,\mathrm{Tr}\left(\delta^+_A J^+ B \wedge J^- B \wedge \eta\right)$$

Thm (B) Assume $M^5 = \mathbb{R}_s \times X^3 \times \mathbb{R}^+_y$, $v = \partial_y$

- $\mathbb{R}_s \times X^3$ is ALE or ALF grav. Instanton
- (A, B) satisfy BCs as before (NPK & vw)

Then $\int_{M^5} d\chi \to 0$.

Cor Then $HW_v(A,B) = 0 \iff DHW_{v,J}(A,B) = 0$

and $HF^\bullet((\mathbb{R}^3; K))$ is fully determined by "decoupled" HW-instantons & "decoupled" KW-sols.

proof idea

regularize $\int_{M^5} d\chi$ by a compact exhaustion of M^5 that respects incidence angle between $\partial_i M^5$ & v and correspondingly the BCs.

Stokes theorem $\rightsquigarrow \sum_i \int_{\partial_i M^5} \chi$

@ $y \to 0$

elliptic regularity $\Rightarrow$ (A,B) polyhomogeneous

i.e. well-behaved asymptotic series in $y^\alpha (\log y)^k$.

Expand around NPK-part at $\mathcal{O}(y^{-1}), \mathcal{O}(R^{-1})$

$$\rightsquigarrow \chi = \begin{cases} \mathcal{O}(y^\varepsilon) \ \mathrm{vol}_{\partial_0 M} & (y \to 0) \\ \mathcal{O}(R^{-1+\varepsilon}) \ \underbrace{\mathrm{vol}_{\partial_K M}}_{\sim R\,dR} & (R \to 0) \end{cases}$$

@ $y \to \infty$

Conj (Nagy-Oliveira '21, B '23)

W^4 ALE or ALF, $KW_\theta(A,\phi) = 0$, finite energy

$\Rightarrow$ A is flat, $\nabla^A \phi = 0 = [\phi \wedge \phi]$

$\rightsquigarrow$ $\mathcal{K} \propto \nabla^A \phi$, $[\phi \wedge \phi] = 0$

(5) Outlook

Generalize He-Mazzeo's results for

$$\text{EBE} \left\{ \begin{array}{l} \mathcal{D}_0 = 0 \\ [\mathcal{D}_i, \mathcal{D}_j] = 0 \\ \sum [\bar{\mathcal{D}}_i, \mathcal{D}_i] = 0 \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Higgs bundles} \\ \text{w/ extra structure} \\ \quad @ K \end{array} \right\}$$

Seems to lead to something that looks like symplectic Khovanov homology.