Harmonic Persistent Homology

Basu-Cox (2021), arxiv: 2105.15170

Problem: PH provides complete top. invariant, but lacks canonical representatives

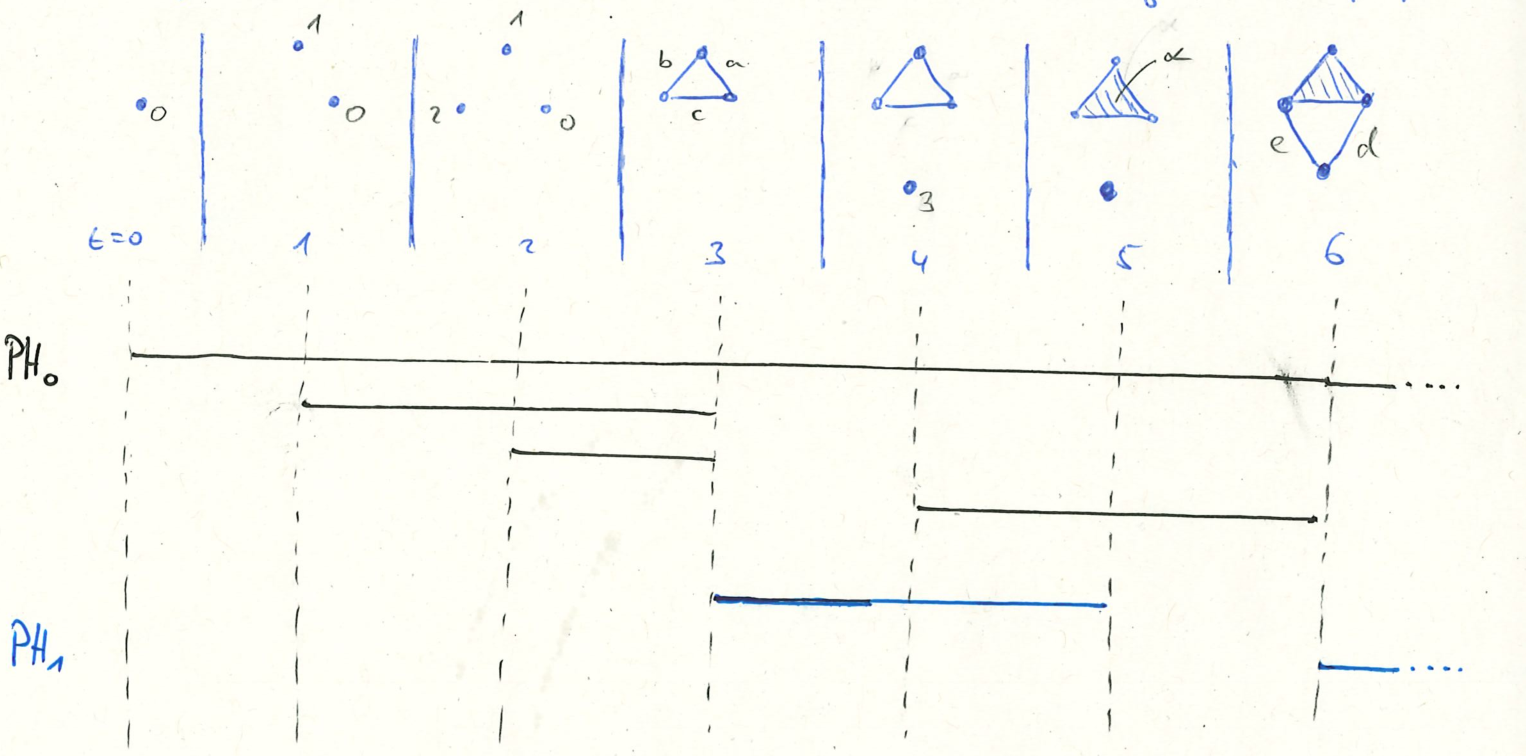
for many applications

Idea: Harmonic PH assigns meaningful, stable, interpretable (?) representatives.

cf. Hodge decomposition theorem

"every homology class has a unique harmonic representative"

Today's goal: understand harmonic PH of the following filtered simpl. cplx. $K_\bullet$



or in formulas:

$$PH_0 = I_{[0,\infty)} \oplus I_{[1,3)} \oplus I_{[2,3)} \oplus I_{[4,6)}$$

$$PH_1 = I_{[3,5)} \oplus I_{[6,\infty)}$$

interval modules

1. A harmonic opening

harmonic functions $\Delta f = 0$ where $\Delta = \partial_x^2 + \partial_y^2 + \dots = d^* d + d d^*$

exterior derivative

adjoint wrt. inner product
$\langle d^* \cdot, \cdot \rangle = \langle \cdot, d \cdot \rangle$

from music via wave equation $\partial_t^2 f = \Delta f$

eigenvalues of Δ are eigenfrequencies and sound harmonic when played together

- equilibrium state, end state, no flux state of wave eqn. or heat flow.
- maximum principle
- mean value property $f(x) = \frac{1}{|S_r|} \int_{S_r(x)} f$

i.e. strong averaging behaviour (keep this in mind)

2. The main theme

K simpl. compl.

$C_p(K) = \bigoplus_{\sigma \in K^{[p]}} \mathbb{R} \cdot \sigma$ p-chains w/ <u>real</u> coefficients

To define Laplacian need an inner product on $C_p(K)$

$\langle \sigma, \eta \rangle = \delta_{\sigma, \eta}$ "standard inner product"

<u>Def</u> codifferential $\langle \partial^* \sigma_p, \eta_{p+1} \rangle = \langle \sigma_p, \partial \eta_{p+1} \rangle$

e.g. $\partial^* v_0 = -(a+c+d)$ st. $\langle \partial^* v_0, a \rangle = -1 = \langle v_0, \underbrace{\partial a}_{v_1 - v_0} \rangle, \dots$ in $C_p(K_6)$

~~Def Laplacian $\Delta =$~~

<u>Note</u>: ∂^* spreads weight from v_0 to its cofaces

all simplices that contain v_0 as a face

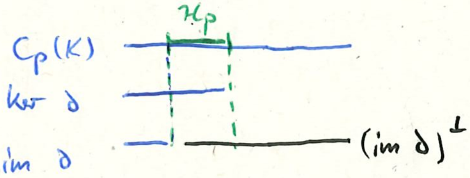
$$\dots \to C_{p+1}(K) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} C_p(K) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} C_{p-1}(K) \to \dots$$

<u>Def.</u> (combinatorial) Laplacian $\Delta = \partial^* \partial + \partial \partial^*$

Thm: Hodge decomposition $C_p(K) = \partial^* C_{p-1}(K) \oplus \underset{\substack{\| \\ \ker \Delta}}{\mathcal{H}_p(K)} \oplus \partial C_{p+1}(K)$.

"curl" harmonic "divergence"

moreover $\mathcal{H}_p(K) \simeq H_p(K,\mathbb{R})$.



Prop. $\mathcal{H}_p(K) = \ker \partial \cap (\operatorname{im} \partial)^\perp$

and $\operatorname{proj}_{(\operatorname{im}\partial)^\perp} : H_p \xrightarrow{\sim} \mathcal{H}_p$ is an iso.

This is the definition proposed by Eckmann (1944).

Example $K_6 =$ 

$\ker \partial = \operatorname{span}\{\underset{\substack{\| \\ x}}{a+b+c}, \underset{\substack{\| \\ y}}{c-d+e}\}$

$\operatorname{im} \partial = \operatorname{span}\{a+b-c\}$

$\ker \partial \cap (\operatorname{im} \partial)^\perp = \operatorname{span}\{\underset{\substack{\| \\ x+3y}}{a+b+2c-3d+3e}\}$

Thm (Basu-Cox)
Harmonic representatives / cycles maximize the (relative) essential content.

Def. $\sigma \in C_p(K)$ is called essential to $z \in H_p(K)$ if all representatives are

$$z = \cancel{}\ c_\sigma \cdot \sigma + \sum_{\eta \in K^{(p)}} c_\eta \cdot \eta \quad , \ c_\sigma \neq 0$$

write $\Sigma(z)$ for the set of essential simplices.

Example • $z \in H_1(K_6)$ has $\Sigma(z) = \{e, d\}$

•  ~~the~~ H_1 class has no essential simplices.

Def. (relative) Essential content of $z \in C_p(K)$ $\|\Sigma(z)\|^2 := \left(\dfrac{\sum_{\sigma \in \Sigma(z)} c_\sigma^2}{\sum_{\sigma \in K^{(p)}} c_\sigma^2} \right)$, ~~any representative~~ $z = \sum_{\sigma \in K^{(p)}} c_\sigma \cdot \sigma$

3. A persistent note

$$\begin{tikzcd}
K_s \arrow[r, hook] \arrow[d] & K_t \arrow[d] \\
H_p(K_s) \arrow[r, "i^{s,t}"] \arrow[d, "\mathrm{proj}_{(\mathrm{im}\,\partial)^\perp}"', "?"] & H_p(K_t) \arrow[d, "\mathrm{proj}_{(\mathrm{im}\,\partial)^\perp}", "\wr"'] \\
\mathcal{H}_p(K_s) \arrow[r, "j^{s,t}"] & \mathcal{H}_p(K_t) \\
\cap\!| & \cap\!| \\
C_p(K_s) & C_p(K_t)
\end{tikzcd}$$

$$I_{[b,d)} = \begin{cases} \mathbb{R} & \text{for } t \in [b,d) \\ 0 & \text{else} \end{cases} \quad + \text{ maps}$$

Interval modules from $\mathrm{im}(i^{s,t})$

$\rightsquigarrow PH(K_\cdot) = \bigoplus I_{[b,d)}$

$\rightsquigarrow P\mathcal{H}(K_\cdot) := \bigoplus J_{[b,d)}$

Interval modules from $\mathrm{im}(j^{s,t})$

$$J_{[b,d)} = \begin{cases} x\cdot\mathbb{R}, & x \in \mathcal{H}_p(K_b) \\ j^{s,t}(x)\cdot\mathbb{R}, & b<t<d \\ 0 & \text{else} \end{cases}$$

w/ $j^{s,t} = \mathrm{proj}_{(\mathrm{im}\,\partial)^\perp}$ projects out additional simplices

Key distinction: harmonic PH is made from harmonic subspaces of $C_p(K_\cdot)$ instead of (abstract) vector spaces.

$$PH_p : (\mathbb{R}, \leq) \to \mathrm{Vect}$$

$$P\mathcal{H}_p : (\mathbb{R}, \leq) \to \mathrm{Gr}\big(C_p(K_\infty)\big) = \coprod_k \mathrm{Gr}_k\big(C_p(K_\infty)\big)$$

Grassmannian / lin. subspaces of dim k

Since the harmonic subspace @ b uniquely determines $J_{[b,d)}$, we identify them.

The harmonic barcode is the set

$$\left\{ (b_i, d_i, J_i) \in \mathbb{R}\times\mathbb{R}\times\mathcal{H}_p(K_{b_i}) \right\} \qquad \text{(Basu–Cox)}$$

and extends the usual barcode by the additional information of a harmonic subspace (1d) and any of its generators is a harmonic representative.

Alternatively, pick a generator $x \in J_i \subseteq \mathcal{H}_p(K_{b_i})$ and consider (b_i, d_i, x_i) as "harmonic bar".

4. The final cadence

To get the harmonic PH of our example $K_\bullet$, we only need to determine the harmonic representatives at the birth time of each homology class.

$$P\mathcal{H}_0 = \operatorname{span}\{v_0\}_{[0,\infty)} \oplus \operatorname{span}\{v_1\}_{[1,3)} \oplus \operatorname{span}\{v_2\}_{[2,3)} \oplus \operatorname{span}\{v_3\}_{[4,6)}$$

$$P\mathcal{H}_1 = \operatorname{span}\{a+b-c\}_{[3,5)} \oplus \operatorname{span}\{a+b+2c-3d+3e\}_{[6,\infty)}$$